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Errors and Misconceptions in Euclidean Geometry Problem Solving Questions: The Case of Grade 12 Learners

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ABSTRACT

Euclidean geometry provides an opportunity for learners to learn argumentation and develop inductive and deductive reasoning. Despite the significance of Euclidean geometry for developing these skills, learner performance in mathematics, particularly geometry, remains a concern in many countries. Thus, the current study examined the nature of learners' errors in Euclidean geometry problem-solving, particularly regarding the theorem for angle at the centre and its applications. Van Heile's theory of geometric thinking and teacher knowledge of error analysis were used as conceptual frameworks to make sense of the nature of learners' errors and misconceptions. Using a participatory action research approach, the study was operationalised by five mathematics teachers from four secondary schools in Motheo district in the Free State Province of South Africa and three academics from two local universities. The study analysed 50 sampled midyear examination scripts of Grade 12 learners from four schools. The findings of this study revealed that most learner errors resulted from concepts on Van Heile's operating Levels 0 and 1, while the questions mainly required Level 3 thinking. The study recommends that teachers determine their learners' level of geometric thinking and integrate this knowledge in their lesson preparations and material development.

KEYWORDS

Euclidean geometry; geometry; geometric thinking; errors; Van Heile`s theory.

INTRODUCTION

Euclidean geometry is regarded as a critical secondary-school mathematics topic that provides an opportunity for learners to acquire argumentation skills and develop inductive and deductive reasoning (Fujita et al., 2010). However, learner performance in mathematics, particularly geometry, remains a concern in many countries. The study on learners' performance in geometry, found that Grade 8 and 9 learners in Japan experienced challenges when constructing geometry proofs (Fujita et al., 2010). Part of the problem is the development in learners of an adequate understanding of geometry concepts, and the development of geometric reasoning (Ali et al.,2014).Whilst there have been attempts to study learners' errors in mathematics, literature suggests a conceptual shift in attitudes about learner errors and misconceptions, from being considered in a negative light, as difficulties learners face when solving mathematics problems, to being understood as providing opportunities for teachers to understand how learners think and conceptualise mathematics content (Ali et al., 2014; Borasi, 1986). As Van Dijk (2006) argues, it is through text and talk that we can gain access to the minds (thinking) of social actors; thus, in the same vein, learners' errors and misconceptions act as an interface to learners' thinking in relation to geometry concepts, which teachers could use as resources for understanding learners' thinking.

While agreeing with the forgoing conception of learners' errors, Brodie (2013) argues that learners' errors should not be seen as problems that should be avoided but, instead, as opportunities for teaching and learning. In her work, which was a data-informed practice improvement project, Brodie (2013) focused on ways teachers can learn to work with learners' errors, and teachers' views of learners' errors. The body of knowledge relating to learners' errors, with a special focus on teacher knowledge, such as the type of error a teacher chooses to deal with, and the way the errors are dealt with have been studied (Gardee, 2015; Sapire et al., 2016).

Despite positive evidence that it is possible to enhance teacher knowledge using learners' errors, including that provided by the studies referred to above, and others, little is known about learners' geometric thinking when they make errors, or their misconceptions when they attempt to solve Euclidean geometry problems (Brodie, 2013). This kind of information is important, because it provides insight into learners' thinking that lead to making errors, and the possible causes of errors, which could, in turn, improve teachers' knowledge and help them develop appropriate corrective measures for teaching and learning Euclidean geometry (Gardee, 2015). One critical aspect of teacher knowledge for teaching mathematics is teachers' ability to analyse learners' incorrect solutions and determine what could have caused this erroneous thinking, with the intention of developing appropriate corrective measures (Ball et al., 2008). The focus of the current paper is on analyse learners 'errors and misconception when solving Euclidean geometry problems. The study examined the nature of learners' errors in Euclidean geometry problem-solving, particularly regarding the theorem for angle at the centre and its applications.

Literature review and conceptual framework

Conceptualisation of learners' errors and misconceptions

The study of learners' errors is nothing new, and it represents a growing body of knowledge. Literature suggests that there are several reasons why learners make errors when they solve mathematics problems. The work of Olivier (1989) helps us to understand that different theoretical lenses influence the way we understand learners' errors. Olivier (1989) provides evidence for his contention, by showing how behaviourist and constructivist perspectives, respectively, would conceptualise learners' errors in different ways. He argues that, from a behaviourist viewpoint, learners' errors and misconceptions are not important; whilst the constructivist perspective postulates that current understanding influences new learning. Thus, teachers' understanding of learners' errors and/or misconceptions is important for ensuring better learning in the future.

As part of the early work in this body of knowledge, particularly in South Africa, Olivier (1989) argues that we should distinguish between slips, errors, and misconceptions. Herholdt and Sapire (2014) who agree with Olivier (2010) and who draw on Ketterlin-Geller and Yovanoff (2009), state clearly that, "slips are random errors in declarative or procedural knowledge, which do not indicate systematic misconceptions or conceptual problems" (p. 23). Thus, studying slips may be less helpful for pedagogical reasons because Olivier (2010) explains that slips are made by both experts and novices, and they are corrected spontaneously. Whilst slips are sporadic, errors are more systematic (Gardee, 2015) and deterministic, and literature suggests that they are not always context bound. Nesher (1987) whilst in agreement with forgoing conception of errors, creates a connection between errors and misconceptions, and claims that errors do not occur randomly, but have their roots in erroneous principles. Nesher (1987) argues, further, that these erroneous principles, which the author terms misconceptions, can explain not only one, but a whole cluster of errors.

Van Hiele's theory of geometric thinking

To gain a conceptual understanding of learners' errors, this paper draws on the work of Dina van Hiele and her husband, Pierre Marie van Hiele, who, in 1984, proposed a theory of geometric thinking (Fuys, 1984). It is generally accepted that the Van Hieles' description of geometric thinking provides the best explanation for the way learners think when they solve geometry problems. This theory posits that there are five levels of geometric thinking, which are presented in Table 1.

Level 0, called recognition or visualisation, refers to learners' reasoning based on pictorial representation of geometric shapes. Thus, when their thinking is on Level 0, learners disregard geometric properties; their reasoning regarding the differences or similarities between two shapes is based on the shapes' pictorial representation (Howse & Howse, 2015). According to Van der Sandt and Nieuwoudt (2005) learners at Level 0 judge geometric figures by appearance alone. Level 1 of Van Hiele's theory of geometric thinking is called analysis or description. At this level, the theory postulates, learners' reasoning about geometric figures is based on describing the figure by its properties (Fuys, 1984). For instance, two geometric figures are similar or different based on their properties, however, the properties of each figure are still seen to be independent (Van Hiele, 1999, Vojkuvkova & Haviger, 2013). At Level 2, ordering or informal deduction, learners recognise relationships between the properties without constraints of pictorial representation. Furthermore, the learner applies logical reasoning using properties of a shape, though not carefully and without reference to sufficient conditions (Fuys, 1984). The third level is deduction, at which learners' reason logically by ordering the properties of geometric concepts and using their definitions to determine sufficient and insufficient conditions for the concept to be true (Burger & Shaughnessy, 1986). It is at deduction level that learners' reason formally by constructing geometry proofs with a series of logical, deductive mathematical statements (Mayberry, 1983). At the fourth level, which signifies rigor, learners are expected to give reasons beyond the deductive series of properties of geometric shapes, and progress to reasoning with deductive axiomatic systematics for geometry, by exploring their similarities, relationships, and differences.

Table 1

Van Hiele's description of geometric thinking

Level	Name of level	Description
0	Recognition/Visualisation	Judgement of geometric figure is by appearance only
1	Analysis/Description	Properties are recognised but not ordered logically
2	Ordering/Informal Deduction	Properties are ordered logically
3	Deduction	Properties are used to make assumptions, which can be proved logically to be true
4	Rigor	Deductive axiomatic systems are used to reason logically than just making use of properties of geometric shapes

Related literature on geometry and Van Hiele's theory of geometric thinking

That geometry forms part of school curricula is nothing new. It helps learners to develop spatial ability (Battista et al., 1982). Geometry is, in fact, one of the basic mathematics skills that is applied in other subjects too, such as engineering drawing (Abdullah & Zakaria, 2013). Literature suggests that geometry helps learners develop and improve their logical reasoning and argumentation; thus, to develop learners' logical and deductive reasoning through geometry problem solving, consideration of the levels of Van Hiele's geometric thinking is critical when compiling lessons (Battista et al., 1982).

The critical nature of geometric thinking is corroborated by the results of a study by Abdullah and Zakaria (2013) on assessment of learners' levels of geometric thinking in transformation geometry. They found that, before their experiment, both the control and treatment groups of their quasi-experiment exhibited low acquisition of levels of Level 2 thinking, and no acquisition of deductive reasoning. Low acquisition of geometric thinking skills confirms the argument that part of the problem is that learners have only partial understanding of constructing mathematics proof (Imamoglu & Togrol, 2015).

This partial understanding is, arguably, noticeable at a lower level of competency regarding deductive reasoning, which is Van Hiele's Level 3. Jones (2000) points to evidence that learners operating at the lower levels of Van Hiele's geometric thinking (Levels 0, 1, 2) is traceable to classroom practices, where learners experience deductive reasoning as being too difficult, particularly in geometry. Mejia-Ramos et al. (2012) building on the model of Yang and Lin (2008) argue that, when assessing learners' comprehension of a proof, another criterium that learners must demonstrate, in addition to meaning making, logical status and logical chaining of its statement, is high-level ideas of the proof, its main components, the methods employed make an argument and ways their proofs can be applied to specific examples.

METHODOLOGY

A participatory action research approach

This study employed participatory action research (PAR) as an approach to operationalise the study's objectives. The origins of PAR as a research approach can be traced to 1944 and the work of Kurt Lewin, who is regarded as the father of action research (Gillis & Jackson, 2002). Lewin subscribes to the philosophy that people are more motivated to work if they are involved in decision-making processes (McNiff & Whitehead, 2011). Central to PAR is its democratic nature, which means there is space for all participants – all are equally important to and worthy of driving the process of research (MacDonald 2012). Thus, the current study reports on the engagement of five mathematics teachers and a university mathematics teacher educator, and their analysis of learners' errors when solving Euclidean geometry problems. The goals of PAR include identifying changes of immediate benefit to research participants, investigating their social problems and finding ways to resolve these problems. Thus, PAR created an opportunity for a dual aim to be pursued through this study. The first aim related to teachers, who were required to develop mandatory subject improvement plans, which would serve as a diagnostic tool to identify the errors learners make on a particular assessment task. Once errors had been identified, teachers had to determine a possible cause for each error, to develop intervention mechanisms to improve learner performance. The researcher's role in the team effort in pursuit of this aim, was to contribute research knowledge and skills, such as drawing on literature to provide a more comprehensive description of learners' errors, beyond everyday knowledge. However, it was equally important that systematic knowledge of literature contributed to addressing everyday challenges that teachers face in the teaching of Euclidean geometry.

Secondly, the researcher's aim was to contribute to the body of knowledge on learners' errors. In so doing, the researcher had to elevate the level of analysis, so that it was more rigorous and systematic, and synthesise theory and practice.

The likelihood of achieving the goals that had been identified was vested in a unique approach to research, which is democratic in nature and values everyone's voice. The critical vision of PAR helps us to understand that power is always with us; thus, what matters is that it is used positively to open spaces for other voices in the process of research. PAR further enables me, as a researcher, to understand the object of study, not only considering scholarly literature, but equally importantly, from the perspective of those people who are experiencing the problem under investigation. My use of the phrase "equally importantly"' is carefully delimited to Habermas' theory of communicative action. Huttunen and Heikkinen (1998) explain that communicative action,

"Means interpersonal communication which is orientated towards mutual understanding and in which other participants are treated as genuine persons, not as objects of manipulation. Actors do not primarily aim at their own success but want to harmonise their action plans with the other participants" (p. 311).

Thus, by harmonising our action, the most compelling and logical argument explaining possible learner errors is accepted. Furthermore, the communicative rationale is reflective and open for dialogue. Using PAR in scholarly work has caught the interest of many scholars over the years; for instance, Mhina (2009) used PAR to find solutions for the problems experienced by women of Maruku village in Tanzania regarding their inability to access and control agricultural land. Udas (1998) helps us to understand that the value of PAR as an approach lies in its methodology, that is, teachers are not objects of research, but participate in research to understand learners' errors better, and to determine what they could do to develop corrective mechanisms.

Participant researchers and reasons for their inclusion

The study was operationalised by five mathematics teachers from four secondary schools in Motheo education district in the Free State province of South Africa, and one academic from a local university. Two teachers, aged 34 and 45, had 11- and 15-years' experience respectively teaching Grade 12 mathematics. The other three teachers were aged 31, 28 and 43 and had three, four- and 10-years' experience of teaching Grade 12 mathematics respectively. Lastly, the academic had been a mathematics teacher educator for four years and had taught Grade 12 mathematics for six years.

When carrying out a PAR study, the success of its approach to research lies in its quest to work with the participants, who become part of research to improve their lives (Mhina, 2009). For instance, the reason for including teachers in this study was to address the problem of learners' poor performance in Grade 12 final examinations. As a result of learners' poor performance, the Department of Basic Education (DBE) had introduced Act 31 of 2007, which requires schools to submit subject improvement plans to the heads of Education in the respective provinces (DBE, 2011). Thus, the participating schools had been classified as underperforming schools by the DBE. The schools invited the local university to help them develop and implement subject improvement plans. The schools' participation in the research process created an opportunity to examine learners' errors and misconceptions and their possible causes and developing appropriate corrective mechanisms.

Data generation and analysis

The sources of data were learners' scripts, a total number of 50 from four schools. The learners' scripts were selected randomly, and each script was analysed by all co-researchers, who then shared their notes during a discussion. The data were categorised and grouped according to the learners' high-level ideas (Mejia-Ramos et al., 2012), which refer to main argument from which all the preceding logical reasoning flowed. For each main argument, percentage frequencies are presented to account for all the data. Furthermore, the themes were generated using the conceptual framework of Van Hiele's levels of geometric thinking.

All participants were informed of the ethical considerations applicable to the study. The study was planned and executed in a manner that would not cause harm to or threaten the lives of the participants. The university's ethical clearance protocol was observed and permission to conduct the study at the participating schools was sought from and granted by the Free State provincial DBE. Participants were asked to sign consent forms and they agreed to be part of the study. Furthermore, they were made aware that they could withdraw from the research project any time they wished to.

FINDINGS

This section presents the data generated by the study. Analysis was done by, first, clustering learners' errors and formulating the themes as presented in Tables 2 and 3. Each table focuses on a specific question and presents learners' thinking during attempts to respond to questions relating to geometry content.

Interlinked visual and analysis level of geometric thinking

Table 2 presents a thematic analysis of learners' errors on the question that assessed learners' competence in completing the theorem statement exhibited by Table 2. A total of 50 learners' scripts were randomly selected for analysis ($n = 50$). A total of 37 scripts ($n_1 = 37$), which is 74% of learners, presented learners' errors and/or misconceptions, whilst $n_2 = 13$ learners' responses to the question were correct. The themes were formulated by grouping similar responses together and presenting the frequencies per error. Deciding whether responses or errors were similar was done after the learners' main ideas had been considered.

Theme 1 errors, which involved 62% of learners' errors, relate to learners' inability to identify the equality relationship between the angle at the centre of a circle and the angle at the circumference of the circle. There is evidence of partial understanding of the relationship between the angle subtended by the arc at the centre and the angle at the circumference. This argument is substantiated by the fact that these learners could identify the angle that is subtended by the arc at the centre correctly, that is, "angle at the circumference". In addition, reference to "equal to" means they used the correct mathematical operational relationship that exits between the two angles. However, what these learners seem to have missed is the proportional relationship between the two angles, which is, one is twice (double) the other, or one is half the other, in magnitude.

Table 2

Summary of learners' errors on completing the theorem statement on determining the relationship between of the angle subtended by a chord or arc at the centre of a circle

Question: The angle subtended by a chord or arc at the centre of a circle is …		
Total number of learners: n = 50		
Total number of errors: $n_1 = 37$		
Total number of accurate responses: $n_2 = 13$		
Theme 1: Angle at the circumference		23 (62%)
equal to the angle at the circumference	7	
equal to twice the angle at the circumference	11	
equal to twice circumference	5	
Theme 2: Angle at the centre		8 (22%)
twice the angle at the centre	8	
Theme 3: Equal to ninety degrees		6 (16%)
90 degrees	1	
Equal to 360 degrees	1	
Twice 360 degrees	3	
180 degrees	1	

Theme 2 presented with errors similar to those of the first theme; however, they differ in that these learners described the proportional relationship between the two angles by using the word "twice". Despite their competence in recognising the proportional relationship between the two angles, they failed to make their statement completely valid by specifying which angle(s) at the circumference of a circle. For the third theme, learners were able to identify the proportional relationship that exists between the two angles; however, they referred to the same angle twice, that is, the angle at the centre.

Van Hiele's levels of geometric thinking are helpful for explaining learners' thinking on the theorem statement in question. Stating the theorem in words and not pictorially could have posed a challenge for learners. In response to the question, learners had to apply the analysis level of geometric thinking to identify the properties, or as the CAPS (DBE, 2011) policy states, "parts of a circle", such that the theorem statement is true. Furthermore, learners had to take a step further, beyond the analysis level of thinking, and apply the informal deduction level of geometric thinking to identify the relationship between the properties. Learners' responses reported in Table 2 show that some of them operated at the analysis level of geometry thinking, because they could correctly identify certain properties of the theorem statement.

Furthermore, referring to literature, there is evidence of learners operating at informal deduction level; this happened when learners' identification of the properties of a geometric concept was not limited to its pictorial representation. Even though learners where able to identify the properties of the theorem, operating at Level 2 could explain their errors, since most of them showed partial knowledge of sufficient properties that would make the statement true.

Visualisation and analysis

This section reports on the results reported in Table 3, which summarises learner errors on proving that PQRN is a cyclic quadrilateral, thematically. Data analysis of learners' errors and/or misconceptions shows that, according to the sampled scripts that were analysed, no learners who operated at Level 3 and 4 were found to have made errors. The forgoing finding corroborates the findings of Alex and Mammen (2016), who found that learners who participated in their study could operate at Level 2, and no higher. Thus, the focus of the analysis is mainly on the first three levels.

Table 3

Summary of learners' errors on proving that PQRN is a cyclic quadrilateral

Total number of learners: n = 50			
Total number of errors: $n_1 = 31$			
Total number of no responses: $n_3 = 6$			
Total number of accurate responses: $n_2 = 13$			
VISUALISATION (THEME 1)			
$\hat{P} = \hat{R}$, $\angle's$ on the same segment	These errors were categorised under visualisation because learners based their argument or conclusions on recognition or visualisation, and not properties and/or logic	7	17 (46%)
$\hat{L} = \hat{K}$, $\angle's$ sub by the same arc		4	
$\hat{Q} = \angle\text{KNM}$, $\angle's$ sub by the same arc		5	
$\angle KNP = \angle KLM$, Ext $\angle'$s of quadrilateral are equal.		2	
ANALYSIS (THEME 2)			
$\hat{N} = \hat{Q}$, Opposite $\angle's$ of a cyclic quad	These errors were categorised under analysis because learners based their arguments or conclusions on properties	5	14(38%)
$\hat{P} = \hat{R}$, Opposite $\angle's$ of a cyclic quad		3	
$\hat{N} + \hat{Q} = 180°$, 4 sides, 2 sides are parallel		3	
All angles in cyclic quad add up to $360°$		1	
$\angle KLM = \angle KNM$, opposite side of parallelogram		1	
PQ = NR, NQ = PR, therefore PQRN is cyclic quad		1	

Visualisation

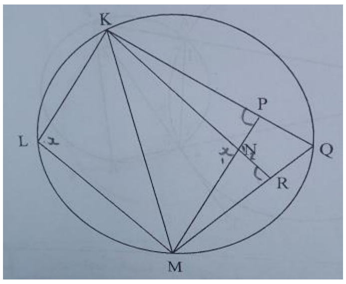
Visualisation refers to geometric reasoning that is limited to the shape(s) or pictorial representation of a geometric concept (Luneta, 2015). For instance, one of the errors under visualisation in Table 3 is that, when learners attempted to prove that PQRN is a cyclic quad, 41% of them argued that $\hat{P} = \hat{R}$, giving as reason that $\hat{P}$ and $\hat{R}$ are angles on the same segment KM. Similarly, 24% of the learners said that $\hat{L} = \hat{K}$, because the two angles are subtended by the same arc KM; and 29% of the learners said that $\hat{Q} = \widehat{N}$, for the same reason, namely, that the angles are subtended by the same arc KM. What is common in all these responses, given by 94% of learners, is that they have a partial understanding, limited by visual reasoning of the theorem (angle subtended by chord, arc or segment are equal). They recognised that all three pairs of angles are subtended by a chord, arc or segment; however, visual reasoning made learners miss an important property, which is that all the angles must be on the circumference of the circle.

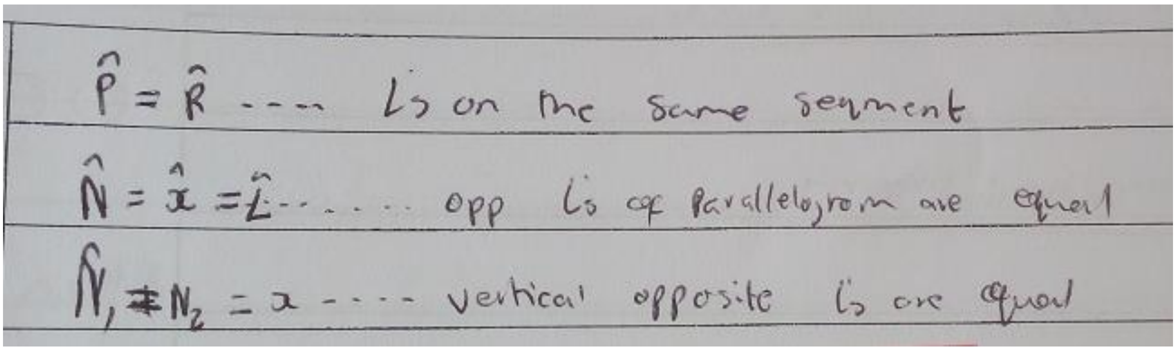
There was considerable evidence of learners operating at visual reasoning level, which also explains a similar learner error, namely that $\angle KNP = \angle KLM$, because the exterior angle is equal to the opposite interior angle of the quadrilateral. Visually, $\angle KNP$ is the exterior opposite angle to interior $\angle KLM$ of a quadrilateral KLMN; however, informal deductive reasoning was needed to make the statement true. This reasoning is that, even though they are exterior and interior angles visually, they will only be equal if, and only if, quadrilateral KLMN is a cyclic quad. Visual reasoning by learners could be the cause of errors in these themes. Abdullah and Zakaria (2013) would have us understand that learners at this level (visualisation) reach conclusions based on recognition of the geometric shape as seen from above.

Figure 1 presents one of the learners' answers. Thabo (pseudonym) is representative of many learners who gave similar responses. Figure 1 is a snapshot of how Thabo marked and labelled the geometric figure provided in the question about proving that PQRN is a cyclic quadrilateral.

Figure 1

Thabo's response to the question: Prove that QPNR is a cyclic quadrilateral





It is evident from Thabo's response that he could operate, at least, at informal deduction level, as he made a substantiated claim that $\angle KNM = \angle KLM$, with the reason drawn from the properties of KLMN, namely, that opposite angles of a parallelogram are congruent. However, some of the errors made by learners such as Thabo show that they operate only at the analysis level of geometric thinking. For instance, Thabo's solution, which offers one example of this erroneous thinking, gave evidence of a partial understanding of angles on the same segment or angles subtended by chord/arc KM. He claims that $\angle KPM = \angle KRM$, giving the reason that these angles are on the same segment. However, what Thabo omitted in his geometric thinking, is to provide sufficient conditions for this statement to hold, which is that the angles must lie on the circumference of the circle.

Analysis

This theme relates mainly to learners being able to move beyond merely describing properties of geometric shapes visually. To respond to the question of proving that PQRN is a cyclic quad, the error most learners made was describing or listing one or more properties of a cyclic quad, instead of proving it. Table 3 presents evidence of learners' knowledge of properties of a cyclic quad. Literature suggests that, if learners have knowledge of properties of geometric shapes, but cannot demonstrate how the properties are linked or related, then learners are operating at the analysis level of geometric thinking. This was evident in Table 3, which shows that learners only cited the properties of a cyclic quadrilateral. Figures 2 and 3 present learners' general responses.

Figure 2

John's response to the question: Prove that QPNR is a cyclic quadrilateral

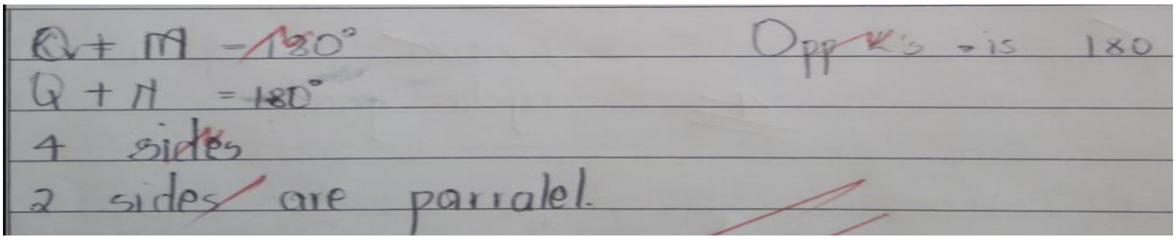
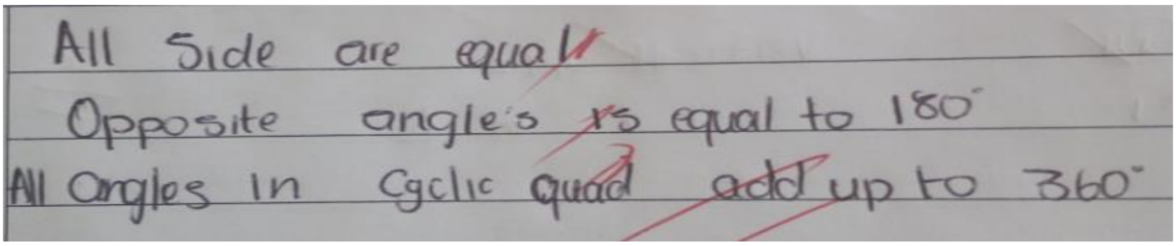


Figure 3

Mpho's response to the question: Prove that QPNR is a cyclic quadrilateral



The responses in Figures 2 and 3 were clearly not slips but were errors. Instead of proving that PQRN is a cyclic quad, the learners cited its properties. Using Van Hiele's theory of geometric thinking, we can infer that these learners were operating at analysis or description level. Part of the reason for my inference concurs with Teppo (1991)'s contention that the reasoning about geometric figures of learners who operate at Level 1 is based on the properties of geometric shapes. Whilst there is evidence of partial understanding of what a cyclic quad is – as some of the cited properties in both Figures 2 and 3 were correct – learners' reasoning seems to be limited to description level. From Figures 2 and 3, and in other learners' responses, there is little or no evidence of learners making informal deductions and attempting to order the properties of acyclic quad logically to develop an argument that QPNR is indeed a cyclic quad. Even more importantly, they do not operate at deduction level either, as there is no evidence of learners presenting logically sufficient condition(s) or properties of a cyclic quad to prove that QPRN is a cyclic quad.

DISCUSSIONS

The aim of this paper was to analyse learners' errors and misconception during their attempts to solve Euclidean geometry problems. The findings of this paper are, thus, focused on learners' geometric thinking that leads to errors.

An analysis of learner errors shows that learners can operate partially at the analysis level of Van Hiele's theory of geometric reasoning, particularly when they must complete a theorem statement. This finding confirms the findings of Solaiman et al. (2017) who found that every learner in a sample of 406 learners operated at, at most, Level 2 of Van Hiele theory.

In addition, the study revealed that many learners operate at the visualisation level of geometric thinking, which caused most of the errors. Most errors occurred because of learners reaching conclusions based of visual representation of a geometric concept, without considering sufficient properties for the geometry concept to be true. For Alex and Mammen (2016) part of the reason why learners perform far below expectation is that they have partial knowledge and skills of geometry. Thus, operating at visualisation level when questions require Level 3 reasoning, means that learners have partial knowledge and skills for Euclidean geometry. This finding suggested an insight into the results of Ali et al. (2014) who found that learners in secondary schools in India performed below the average expectation on questions related to geometry. Part of their problem, as in the current study, is that most geometry questions require learners to operate mainly at Level 3, which is deductive reasoning. However, the current study found no evidence of learners making errors and operating at this level. Adolphus (2011) concurs, but suggests that part of the problem is partial understanding and/or acquisition of the foundational knowledge required, while the results of the current study confirm the findings of Ali et al. (2014) and Adolphus (2011) this study goes further, and provides important specifics about the nature of the errors learners make in response to questions about the theorem, *an angle subtended by arc at the centre is equal to twice the angle on the alternating segment subtended by the same arc*. Learners stated the theorem partially, as they were generally unable to recognise the proportional relationships between the angles.

Not only does the current study confirm learners' weaknesses as captured in the literature, it also investigated learners' responses to determine what learners do possess knowledge about. The study found that the majority of learners know that the angle subtended by the arc at the centre relates to the angle at the circumference, and that opposite angles of cyclic quad are supplementary – even learners operating at Levels 0, 1 and 2. The significance of this finding lies in its pedagogical value – this finding provides practitioners who participated in this study with information on content-specific errors, and on what learners know, which can be useful for suggesting corrective measures or developing teaching strategies (Gardee, 2015).

Conclusion and Recommendations

This study was limited to four schools, of which the participants formed a cluster of mathematics teachers. While the results of this study may be extended to other learners and teachers in similar contexts, it is important to generalise with caution, due to epistemological complexities. The findings of this study reveal that most learner errors are caused by their thinking being on Levels 0 and 1, while the questions require Level 3 thinking. Thus, when they teach geometry, teachers need to determine their learners' level of geometric thinking and integrate it in their lesson preparations and material development. Furthermore, the paper recommends a follow-up study with a larger sample or multiple case studies, which could capture the epistemological variations and complexities and inform pedagogical strategies and knowledge for teaching Euclidean geometry in more diverse educational contexts.

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