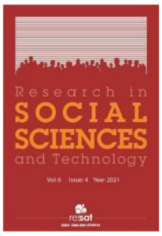

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An Exploration of Grade 12 Learners' Misconceptions on Solving Calculus Problem: A Case of Limits

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ABSTRACT

This case study explored Grade 12 learners' misconceptions in limits of functions. The study aimed at understanding problems that learners face in learning the concepts of limit by analyzing misconceptions that learners have and identifying the possible sources of these misconceptions to take remedial action. An exploratory research design was adopted, and purposive sampling employed to select 35 Grade 12 learners (21 females and 14 males) who wrote a test on limits. From the 35 learners who wrote the test, five learners were interviewed guided by their written responses. Theoretically, the study was guided by the constructivism theory and used a systematic error conceptual framework to categories the learners` misconceptions according to extrinsic and intrinsic calculus misconceptions. The study sought to answer the questions: (a) What type of misconceptions do Grade 12 learners exhibit in responding to limit questions? (b) What are the possible sources of these misconceptions? The study found that Learners exhibited misconceptions on the limit concept and related symbolism. Learners who had a flawed understanding of algebraic concepts struggled to apply limits correctly. The weak foundation on algebraic skills impacted negatively on their learning of new concepts on limits. The study recommends that, educators should always check and make sure that learners have appropriate prior knowledge before the teaching of new concepts in calculus. It is recommended that that teaching and learning approaches need to be adjusted to give equal weight to both the procedural and the conceptual understanding of mathematical knowledge in learners.

KEYWORDS

Differential calculus; extrinsic concepts; Grade 12; intrinsic concepts; misconceptions.

INTRODUCTION

Mathematics is one subject that occupies a strategic place in the South African secondary school curriculum, and one of the key topics taught at Grade 12 level is calculus. Several studies established that learners had challenges in understanding calculus where their performance was poor and there was a tendency to emphasize procedures at the expense of conceptual understanding in the mathematics classroom (Amit & Vinner, 1990; Dlamini et al., 2017; Luneta & Makonye, 2010). The focus on procedural knowledge tends to create learners who do not value the conceptual part of calculus and elect to focus only on the computational aspects, thus making the teaching of calculus lose its essence (Jones et al., 2017; Thompson, 1992). Thompson (1992) argues that learners' errors and misconceptions in calculus may result from educators who focus on teaching mathematical rules, algorithms, and procedures at the expense of developing a conceptual understanding of calculus concepts. The diagnostic reports published by the Department of Basic Education show that learners continue to have problems in the following areas in calculus: The evaluation of limits, algebraic manipulations; graphical application of calculus; and the use of calculus in optimization (DBE, 2015; 2018).

Objectives and research questions:

The study was guided by the following objective:

- To identify some of the systematic errors that Grade 12 learners exhibit in responding to limits questions.
- To establish some of the possible sources of these misconceptions.
- Emanating for the research objectives, the study sought to answer the following research questions.
- What type of misconceptions do Grade 12 learners exhibit in responding to limits questions?
- What are the possible sources of these misconceptions?

LITERATURE REVIEW

Mathematical learning and mathematical knowledge

Researchers have put forward different viewpoints on what constitutes mathematics learning and how best the subject should be taught. According to Skemp (1976), there are two levels of understanding mathematics, termed *relational* and *instrumental* understanding of mathematics. Those who ascribe to the relational understanding of mathematics believe that the correct way of learning mathematics is to know both what to do and why. Relational understanding is about knowing the different mathematical concepts as well as knowing how they are connected. Those who ascribe to the instrumental understanding of mathematics maintain that learning mathematics should essentially be about learning the mathematical rules and how they are applied in computations. Little effort is made in justifying why the mathematical rules work in the different computations or areas of application. Concerning calculus, when one has an instrumental understanding of the concepts, then they cannot justify why calculus techniques work in those areas where they are applied (Jones et al., 2017). There is a possibility that the learner errors and misconceptions in calculus occur because these learners have only mastered an instrumental understanding of calculus concepts but could be struggling with relating these various concepts to each other.

Hiebert and Lefevre (1986) propose two forms of mathematical knowledge that a learner can have, which they referred to as procedural and conceptual knowledge. They argue that when a learner has conceptual mathematical knowledge, they can generalize and connect the different concepts. However, when one only has procedural knowledge, they only know *how* to carry out a mathematical task but are unable to explain *why.* They argue that teaching which emphasizes drill and practice leads to procedural knowledge since the emphasis is on speed and efficiency (Baroody, 2003; Hallett et al., 2010). In solving limits problems, some of the errors and misconceptions reflect the learners' failure to build procedures based on conceptual knowledge. Research by Machaba (2018), established the existence of a bias in South African teachers towards focusing on mastery of procedures and formulae at the expense of a conceptual understanding of mathematical concepts. It may be that due to this kind of teaching, where the emphasis is placed on following rules and procedures, learners end up with a limited understanding of the concepts being taught; hence errors and misconceptions are reflected in their solutions of limits questions.

While there is a consensus that learners need both procedural and conceptual knowledge, there are different views on which knowledge should come first during mathematics teaching. Orton (1983) and Vinner (1989) attributed the problems of calculus teaching to the emphasis on the teaching of calculus procedures while conceptual knowledge is neglected or not getting equal attention.

Another area that impacts learner errors and misconceptions are the teaching strategies adopted by some mathematics educators. Shulman (1986) and Usiskin (2003) argue that some of the errors and misconceptions in mathematical concepts result from the teaching that learners receive. According to Usiskin (2003, p. 86), good mathematics teachers should know a great deal of what he calls “teachers’” mathematics. Shulman (1986) similarly refer to this mathematics content for the teacher, which in essence refers to having an in-depth knowledge of the mathematical concepts coupled with the requisite pedagogical knowledge. A clear distinction is made between content knowledge and pedagogical knowledge, where the former refers to just being competent in handling mathematics concepts, procedures, and problem-solving strategies. The latter refers to the possession of expertise on how to teach content in ways that make it easy for the learners to understand. It further comprises an understanding of what makes the learning of topics easy or difficult (Shulman 1986). Carpenter (1988) also expresses similar views that effective teachers consider the knowledge of related concepts and procedures that learners have already assimilated before they are taught new topics, the likely challenges about the topic they may have developed, and the stages of understanding they are likely to pass through. This type of expertise in teaching methodology is critical for the effective teaching of limits.

Differentiating Errors from Misconceptions on calculus learning

Researchers generally agree on two main categories of errors: unsystematic and systematic errors. According to Khazanov (2008), unsystematic errors are simple once-off incorrect responses that learners can easily rectify on their own once they become aware of them. Examples of unsystematic errors are computational errors and errors due to carelessness. Computational errors result from incorrect addition, subtraction, multiplication, or division. A single mistake by a leaner in a problem that requires multiple steps has a ripple effect, eventually resulting in a wrong final solution. Careless errors occur mainly due to learner inattention or rushing through given work, resulting in incorrectly copying a problem or dropping a negative sign or entering the wrong number into their calculator (Veloo et al., 2015).

The focus of this research is on systematic errors. Systematic errors are consistently repeated over time whereby learners tend to give incorrect responses that are methodically constructed each time they are faced with a particular mathematical problem. According to Nesher (1987), systematic errors have two major characteristics, they are persistent and pervasive, and learners repeat the same mistakes throughout a variety of contexts. Systematic errors are indicative of flawed thought processes; hence they are regarded as misconceptions (Green et al., 2008; Nesher, 1987; Riccomini, 2005). When learners' thought processes are flawed, they keep repeating the same mistakes because they appear to be making sense in their constructions. Because of the perceived sensibleness of the incorrect solutions, these misconceptions can be difficult to correct even when appropriate corrective instruction is implemented (Smith et al., 1993). Nesher (1987), also share similar views that learners' misconceptions often persist and do not easily respond to remedial instruction where one has been designed. Because misconceptions can detract from knowledge acquisition, especially in mathematics; it becomes imperative for researchers to focus and try to understand the root causes of these persistent learner errors better so that educators may be able to assist the learners in acquiring the desired mathematical competencies. While it is easy to identify errors in the learners' written work or in discussions with them, it requires more critical observation in the case of misconceptions that may be camouflaged by correct answers when these answers were obtained by accident (Smith, DiSessa & Roschelle, 1993). Therefore, it is imperative for teachers to be more observant to determine whether the correct answers that learners are giving are not accidental but are the product of correct mathematical reasoning.

Borasi (1987) places systematic errors in several categories as follows:

- According to Borasi (1987), *conceptual errors* arise when learners misunderstand the underlying concepts and use an inappropriate logic to solve a given mathematical problem. A major characteristic of conceptual learner errors is that the learner can make all the correct mathematical computations, but still end up with an incorrect final solution. For example

when a learner is required to determine the turning point of $f(x) = x^3 - x^2 - 5x - 3$, he proceeds to find $f^/(x) = 3x^2 - 2x - 5$ and solves $3x^2 - 2x - 5 = 0$ to get $x = \frac{5}{3}$ or $x = -1$. For
$x = \frac{5}{3}$, $f^/\left(\frac{5}{3}\right) = 0$ and the turning point is given as ($\frac{5}{3}$; 0) where *(x; $f^/(x)$)* is used in place of *(x; f (x))* which is point (5/3; - 256/27). While the computation of $f^/\left(\frac{5}{3}\right) = 0$ is correct it is not addressing the question at hand due to conceptual errors about the value of a derivative at a given point and the coordinates of the point itself.

- *Generalization or transfer errors* refer to the use of techniques or procedures learned in the past in solving new problems without realizing the limitations of those techniques or procedures. For example, while $\sqrt{xy} = \sqrt{x}.\sqrt{y}$, it would be incorrect to assume that $\frac{d}{dx}$ (x^3+1) (*x*-2) = $\frac{d}{dx}$(x^3+1). $\frac{d}{dx}$(*x*-2).
- *Ignorance of rule restrictions* is like overgeneralization. Learners are unable to appreciate the restrictions of certain mathematical structures, and consequently apply a rule in contexts where it is inapplicable (Borasi, 1987). In the context of calculus, given *f(x)*= $\frac{\sqrt{x}-4}{\sqrt{x}}$, this type of error arises when learners incorrectly switch from the square root sign (√) to exponents and write *f(x)* = $\frac{(x-4)^{1/2}}{x^{1/2}}$ incorrectly applying square root sign (√) to *(x-4)*. Hence the learner will have difficulties in finding $f^/(x)$ correctly in the new expression.
- According to Borasi (1987), the incomplete *application of rules* arises in cases where learners fail to learn more complex types of structures opting for the use of relatively simple rules and hoping to succeed with effective communication. In calculus when are learners are required to find the equation of tangent to a given curve at a specific point, for example $f(x) = x^3 - x^2 - 5x - 3$ at *x* = 1, some learners find the correct gradient using $f^/(1)$, but then use $(1; f^/(1))$ as the point of contact to find the equation of the tangent. Here learners fail to distinguish f(x) from. $f^/(x)$
- Another error, cited by Borasi (1987), is learners forming the *wrong hypothesis* about concepts and proceeding to use these false hypotheses in the learning of new concepts. A typical example of this error is when learners are requested to find the *x* and *y* intercepts of $f(x) = x^3 - x^2 - 5x - 3$; some may write $f(x) = x^3 - x^2 - 5x - 3 = 3x^2 - 2x - 5 = 0$. Here the learners confuse intercepts and use of derivatives to find turning points, and misuse = signs, thus mixing up different concepts as they proceed with their calculations.

Categories of Misconceptions in calculus

Over and above the misconceptions discussed by Borasi (1987), Nishimori (2005) chooses to view the misconceptions in terms of how they relate to calculus, using two broad categories intrinsic and *calculus-intrinsic* and *extrinsic.* Calculus-intrinsic concepts are concepts within calculus that are misunderstood, whereas extrinsic concepts are concepts that are preceding calculus and act as prior knowledge that must be learnt and understood before calculus concepts could be understood. Judson and Nishimori (2005) argue that students' difficulties in calculus can be traced to other learning difficulties they experience in dealing with related topics such as functions, graphs, inequalities, summation, and algebra. Their study observed that many students struggled with understanding function concepts, which in turn led to learner errors when they encountered problems involving applying limits of functions due to the interrelationship between functions and calculus. More research confirms that when students have a shallow understanding of the function concept, they experience many difficulties with calculus concepts once they fail to see the relationship between functions and calculus and approach them as isolated concepts. When the concepts are treated in isolation, the net effect is that students tend to adopt and retain incorrect mathematical constructs (Cooley et al., 2007; Orton, 1983; Selden, Selden & Mason, 1994). As a result, there could be over-generalization or transfer errors. Usiskin (2003) argues in favor of introducing students to such concepts as inequalities, summation, and algebra at an early stage. This background knowledge will provide the necessary foundation for students to learn calculus as these concepts are not isolated are interlinked.

According to Morris (1999), another root cause of misconceptions is unsuccessful teaching strategies. He argues that one of the problems for students learning calculus is mathematics educators' tendency to emphasize equipping learners with procedural knowledge grounded in algebra. Consequently, the learners also have difficulties comprehending the importance of the conceptual part of calculus and hence neglect it and consider only the computational part, thereby losing the essence of learning calculus (Bezuidenhout, 2001; Davis & Vinner, 1986). Aspinwall and Miller (1997), in their studies also made the same observation that students regard mastery of the computational skills as the main objective of learning calculus and this approach creates students whose understanding of the calculus concept is shallow.

Further research has found that in the process of solving calculus problems, students fail to utilize all the given information, opting to selectively utilize only part of the information which they consider relevant, thereby ignoring those other parts that would be indispensable in the successful resolution of the problem (Carlson et al., 2003).

The present research focuses on the learning difficulties in limits of functions, learner errors and related misconceptions of mathematics learners in the Limpopo province, South Africa. For the effective analysis of the research findings, the errors will be coded according to the different categories explored above.

Theoretical framework

The constructivist theory of learning forms the foundation for this research. According to this theory, the learner is not treated as an empty tin which absorbs knowledge directly from life experiences or from teaching only in a passive way (Krahenbuhl, 2016). The learner actively participates in the construction of their own knowledge in a process where they construct new knowledge by utilizing what they already know and relating it to what is new so that new ideas assume meanings concerning what is already known by the learner (Shuard, 19861; Wheatley, 1991). Thus, in learning mathematics, learners actively participate in constructing their own mathematical knowledge, using their previous personal experiences as a reference point, developing their ways of thinking as they gain new experience, and using the new experiences to build on and expand their knowledge base.

Table 1.

A Tabular representation of the various types of envisaged misconceptions and their sources

Type of Systematic errors	Category of misconception	Sources
• conceptual errors. • generalization or transfer errors. • ignorance of rule restrictions. • incomplete application of rules • wrong hypothesis used to learn new concepts.	<u>Category a</u> Extrinsic calculus misconceptions	Errors which arose due to knowledge deficiencies in the following *extrinsic calculus concepts*: • functions and related subfunctions such as domain and range. • representations of functions through graphs, rules, tables, and arrow diagrams. • geometry and measurement. • algebra. • inequalities. • summation,
	<u>Category b</u> Intrinsic calculus misconceptions	• Errors which arose due to learners having difficulties with the following ***intrinsic calculus concepts:*** • limits – definition, related symbolism, and graphical representation; • functions – continuity and discontinuity of functions at given points, graphing of functions. • derivatives – definition and related symbolism, relationship between 1st and 2nd derivative, geometric meaning of 2nd derivative

According to constructivist theorist, in all instances, learners construct knowledge in their own different ways, regardless of the teaching methods employed (Krahenbuhl,2016). Even where direct teaching is employed, and students are told mathematical facts or ideas, they do not absorb the ideas exactly as they are taught, but they impose their own interpretation and attach meaning to these ideas through the lens of their own existing knowledge (Krahenbuhl,2016). The fact that the existing knowledge is unique for each learner implies that they are constructing their own knowledge (Cobb, Yackel &Wood, 1992).

Systematic errors are believed to emanate from misconceptions, where learners have constructed incorrect conceptualizations of required mathematical knowledge but are unaware that although it makes sense to them, it conflicts with conventional mathematical knowledge (Nesher, 1987; Smith, DiSessa & Roschelle, 1993). According to Swan (2001), errors can be embraced as providing insights into the learners' thought processes and engagement points. Borasi (1987) also concur that errors should constitute a normal part of both learners and educators' learning process. Because errors seem reasonable and make sense to the person making the error, they are essential in helping teachers engage with learners' current knowledge. Through learners' errors, teachers can appreciate that learners are reasonable thinkers (Ball & Bass, 2003). When teachers strive to understand the sources of learner errors and value learners' thinking, they will be better positioned to utilize what learners already know in helping them construct new knowledge.

The table below summarizes the envisaged misconceptions and their sources from the above literature review. While there were nonsystematic errors, the focus of this study was on misconceptions which are systematic errors. The table shows five types of systematic errors highlighted by Borasi (1987) and the two broad categories of extrinsic and intrinsic calculus concepts which might be underpinning the misconceptions (Judson & Nishimori, 2005). The study focused on these misconceptions and their sources as reflected in Table 1.

METHODOLOGY

This qualitative case study was located within the interpretivist paradigm and took naturalistic methodology. A case study is defined as a "qualitative approach in which the investigator explores a bounded system (a case) over time through detailed, in-depth data collection involving multiple sources of information" (Creswell et al., 2007, p. 245). The study focused on a class of Grade 12 mathematics learners at one school in the Limpopo province. The Grade 12 learners were purposely selected. In purposive sampling, the researcher had the prerogative of deciding what needs to be known and identified the candidates who could and were willing to provide the required information by virtue of their knowledge or experience (Bernard, 2002). A calculus test that focused on limits was administered to 35 grade 12 mathematics students (21 females and 14 males) who consented to participate in the study. The learners were selected because they were taught by one of the researchers and their syllabus contained a section of limits. From the 35 students, 5 students were interviewed guided by their written responses to the test.

Table 2.
Types of errors and categories of misconceptions

Type of Systematic Errors	Category of misconception	Sources
• conceptual errors; **s1** • generalisation or transfer errors; **s2** • ignorance of rule restrictions; **s3** • incomplete application of rules; **s4**	Category a Extrinsic calculus misconceptions; **(ce)**	Errors which arose due to knowledge deficiencies in extrinsic Calculus concepts • functions and related subfunctions such as domain and range. • representations of functions through graphs, rules, tables, and arrow diagrams. • geometry and measurement, algebra, inequalities, summation
• wrong hypothesis used to learn new concepts; **s5.**	Category b Intrinsic calculus misconceptions. **(ci)**	Errors which arose due to learners having difficulties with intrinsic calculus concepts: • limits – definition, related symbolism, and graphical representation; • functions – continuity and discontinuity of functions at given points, graphing of functions. • derivatives – definition and related symbolism, relationship between 1st and 2nd derivative, geometric meaning of 2nd derivative.

Adopted from Borasi (1987) and Judson and Nishimori (2005)

The interview's primary objective was to allow the learners to explain their understanding of the questions and their interpretations and insights into the topic of limits of functions. The responses to interviews provided additional information to the researcher to better understand what was in the learners' minds and the likely sources of errors. An exploratory research design was adopted to observe learner errors and how these errors were interlinked to their misconceptions on limits of functions. Exploratory research helps researchers gain new insights, unearth new ideas, and create knowledge of the phenomenon (Burns & Grove; 2001).

For trustworthiness of the interview questionnaires, two mathematics teachers and three mathematics education researchers reviewed the questions for clarity and specificity. The content validity of the test questions was assured by using the questions in the DBE CAPS document (DBE, 2012). Ethical considerations were made when gathering data. The results of this study are reported using only pseudonyms. Data were analyzed thematically, and all qualitative data were coded to examine common themes and patterns (Maxwell & Miller, 2018). For the written test, the responses to each question were analyzed, and learners were grouped according to the type of errors that they made in each of the questions in the research instrument. Errors were coded according to the similarities of the errors and the whole range of other error categories identified in the literature review. For example, s1 was used for conceptual errors, s2 for generalization or transfer errors, up to s5 for the wrong hypothesis used to learn new concepts. The code ci was used for intrinsic calculus misconceptions, and the code ce was used for extrinsic calculus misconceptions. A table such as the one below was developed in the process of analyzing and categorizing the collected data.

RESULTS and DISCUSSIONS

The Calculus test was administered to a class of 35 Grade 12 learners at a secondary school in Limpopo province, South Africa. After marking, the performance of all 35 learners was summarized through tables. The sample answers from scripts were also presented for purposes of clarification of observed errors and misconceptions. In addition, five learners, L8, L9, L10, L11 and L12, were interviewed in relation to specific misconceptions displayed in their responses to specific questions. The interview responses were analyzed together with evidence from written scripts. The interview questions were intended to serve two purposes, firstly to enable the researcher to better understand the learner's thinking processes and secondly to address and rectify the misconceptions.

The data presentation and analysis attempted to provide answers to these questions that were set up at the beginning of the research project.

(i) What type of errors and misconceptions do Grade 12 learners exhibit in responding to differential calculus questions?

(ii) What are the possible sources of these misconceptions?

Data presentation and analysis

The following table is an itemized summary of learner performance in the written test (See Appendix), where written answers from learners are categorized as g= *good* for correct answers, f=*fair* for answers that are partially correct and w=*weak* for wrong answers.

Table 3.

Item Analysis of learner responses to test questions

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
1ai	g	w	w	w	w	w	w	w	w	g	w	w	f	g	w	f	w	g
1aii	g	w	w	w	w	w	w	w	w	g	w	w	f	g	w	f	w	g
1bi	g	w	w	w	w	g	g	g	g	g	w	w	w	g	w	w	w	f
1bii	f	w	w	w	w	f	f	f	f	f	w	f	w	w	w	f	w	f
1ci	w	w	w	w	w	g	g	f	f	f	g	w	g	w	w	w	w	f
1cii	g	w	w	g	w	w	w	g	w	f	f	g	f	g	w	w	f	w
1ciii	w	w	w	w	w	w	w	w	w	w	w	w	w	f	w	w	w	w
	19	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	
1ai	w	w	w	w	w	f	w	w	f	w	f	w	w	w	w	w	w	
1aii	w	w	w	w	w	f	w	w	g	w	f	w	w	w	w	w	w	
1bi	g	w	w	w	w	w	w	g	w	w	w	w	w	w	w	w	f	
1bii	f	w	w	w	w	w	w	f	w	w	w	w	w	w	w	w	w	
1ci	w	w	w	w	w	w	g	w	g	w	w	w	w	w	w	w	w	
1cii	w	g	w	w	w	w	w	g	w	w	f	w	w	w	w	w	w	
1ciii	w	f	w	w	w	w	w	w	w	w	w	w	w	w	w	w	w	

g =good; f=fair; w=weak

The focus of Question 1 was on assessing the learners' knowledge of the limit concept and related symbolism, the existence or nonexistence of limit and the idea of a limit as a function. The learners' knowledge of limits is important as it acts as prior knowledge for the study of calculus concepts, all of which are defined in terms of limits. *Sub question 1(a),* which required learners to explain in words the meaning of these statements (i) $\lim_{x \to 0} \frac{x^2 - 3x}{x} = -3$ and (ii) $\frac{x^2-4}{x-2} \to 4$ as $x \to 2$ focuses on the learners' understanding of limits and related symbolism. The purpose of the test item is to check if the learners have the correct concept image for the limit concept.

Table 4.

Analysis of results per test item in terms of percentages

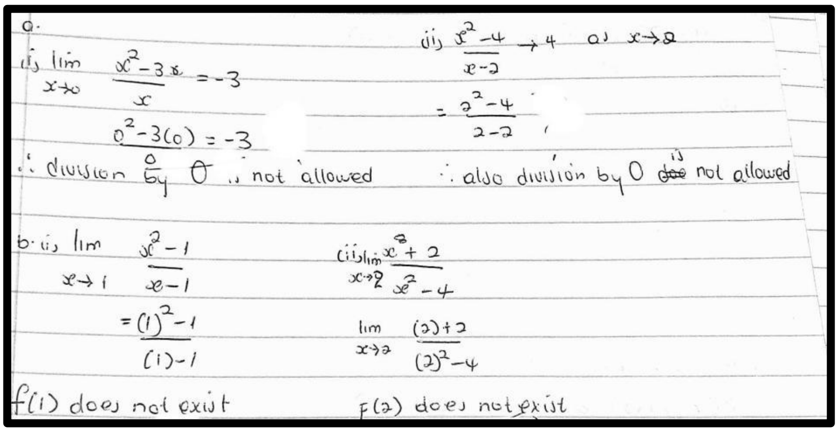
Test item	No. of correct answers	**%** of correct answers	No. of partially correct answers	**%** of partially correct answers	No. of correct answers	**%** of incorrect answers
1a i	4	11	5	15	26	74
1a ii	5	14	4	12	26	74
1b i	9	26	2	5	24	69
1b ii	0	0	11	31	24	69
1c i	6	17	4	12	25	71
1c ii	7	20	5	14	23	66
1c iii	0	0	2	6	33	94

This question exposed several learner misconceptions on the limit concept**.** Table 4 shows that 74% of the learners could not explain the symbolism used in 1a (i)and (ii), 69% could not find the limits of the given functions in 1b (i) and (ii) and 94% could not use the graph to find limits using the graph in 1c(iii).

Several learners in this group of twenty learners, (ci1) (20), interpreted $\lim\limits_{x\to 0}$ to mean substitute x by 0 and by 2 in *x*→ 2. For example, L2 in this group assumed that $\lim\limits_{x\to a}$ meant that where they see *x,* it must be removed and substituted by *a* to get answer for limit. For these learners, $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$ does not exist since $\frac{2^2-4}{2-2}$ is undefined or division by zero is not allowed.

Figure 1

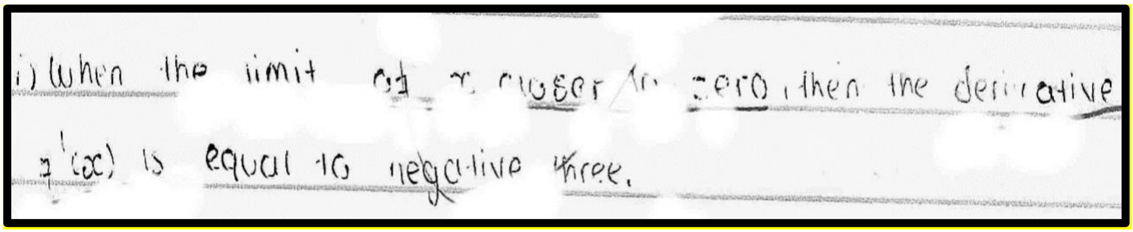
L2's solution to question 1(a) & (b)



L2 concluded that firstly the statements i) $\lim_{x \to 0} \frac{x^2 - 3x}{x}$ = - 3 and (ii) $\frac{x^2-4}{x-2} \to 4$ as $x \to 2$, were mathematically incorrect because in both cases one had to divide by zero, which is not allowed. This showed that the learner did not understand the concept of a limit. Secondly, L2 concluded that f(1) and f(2) did not exist in $lim_{x \to 1} \frac{x^2-1}{x-1}$ and $\lim_{x \to 2} \frac{x+2}{x^2-4}$ respectively, and by implication he was saying that the limits cannot exist if the functions are not defined at the given points. This was a calculus intrinsic misconception since $lim_{x \to 1} \frac{x^2-1}{x-1}$ = $lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$=$\lim_{x \to 1} x + 1 = 1 + 1 = 2$, but f(1) is undefined. However, the limit can exist when the function is undefined at a particular point. Hence, L2 committed a conceptual error, s1, demonstrating a misunderstanding of the limit concept. Another learner, L12 says when the limit gets closer to zero, the derivative is equal to negative.

Figure 2.

L12's solution to question 1(a) & (b)


i) when the limit of x closer to zero, then the derivative f'(x) is equal to negative three.

The solution of L12 shows another conceptual error, s1, which arose from the learners having difficulties with the concepts of a limit and a derivative of a function. Referring to the limit of a function as the derivative was not correct since the derivatives of each of the given functions is different from their limit. It was therefore an intrinsic calculus misconception on the learner's part to think that the limit of a function is synonymous with its derivative. Both the limit and derivative concepts are part of the content being learnt during the teaching and learning of calculus, hence the misconceptions displayed by learners here had their root source in these intrinsic calculus concepts, which learners were struggling to understand.

Sub Question 1(b) on finding the limits where they exist of (i) $\lim_{x \to 1} \frac{x^2-1}{x-1}$ and (ii) $\lim_{x \to 2} \frac{x+2}{x^2-4}$ was designed to check if learners could distinguish the value of a function at a given point from the limit of the function with reference to the given point and their ability to use related limit symbolism.

One major challenge observed in the scripts of learners were flaws in their algebraic knowledge. The samples of learners L29 and L20 are representative of this group of five learners, (ce4.1) (5), who exhibited extrinsic calculus misconceptions by failing to find limits because they had problems with the concept of factorization. The factorization of algebraic expressions is done in lower grades and should form part of the prior knowledge for calculus, hence the learner misconceptions displayed here are resulting from a flawed understanding of extrinsic calculus concepts.

Figure 3.

L29's solution to question 1(b)

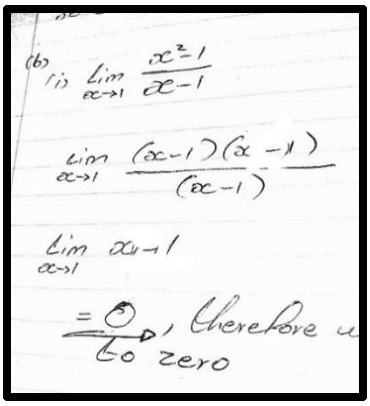
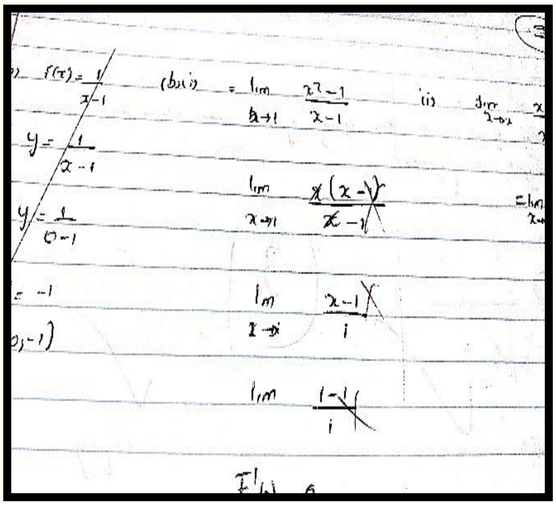


Figure 4.

L20's solution to question 1(b)



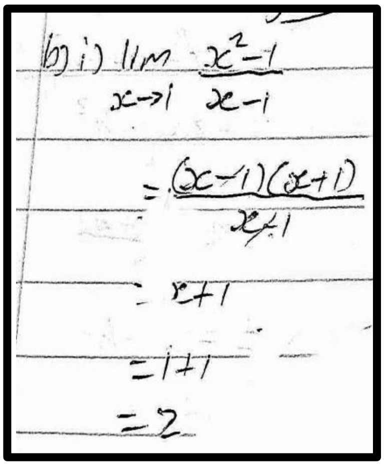
In Figure 3, L29, gave the incorrect factors $(x-1)(x-1)$ as factors of x^2-1 which appears to emanate from the learner's failure to master the concept of factorising a difference of two squares, $x^2-1=(x-1)(x+1)$. This was a conceptual error (s1). In Figure 4, L20, used $x^2-1= \ x(x-1)$, where there is an ignorance of rule restriction error **(s3)** as this learner failed to appreciate that $x(x-1)=x^2-x$ when brackets are removed

Another general observation for the group was that of learners making generalization or transfer errors by using inappropriate procedures learnt earlier to address the new concept of a limit. This group of nine learners, (ci2) (9), was constituted of learners who committed procedural errors of the form $\lim\limits_{x\to1}\frac{x^2-1}{x-1}$ = $\frac{(x+1)(x-1)}{x-1}$ = $x+1$, as L12 did by omitting $\lim\limits_{x\to1}$ during working or including $\lim\limits_{x\to1}$ where it was inappropriate, like writing $\lim\limits_{x\to1}$ x+1 **=** $\lim\limits_{x\to1}1+1=2$. These learners did not appreciate the significance of $\lim\limits_{x\to1}$ in $\lim\limits_{x\to1}\frac{x^2-1}{x-1}$.

Figure 5 clearly illustrative of the impact of a poor understanding of limits. Failure to utilize the appropriate symbolism in carrying out a procedure was indicative of a view held by learners that this symbolism is unnecessary as it does not change the final answer. Learners in this group focused on the answer and lost sight of the significance of the mathematical precision required in using notation and symbolism of limits to ensure that what is written is conceptually correct.

Figure 5.

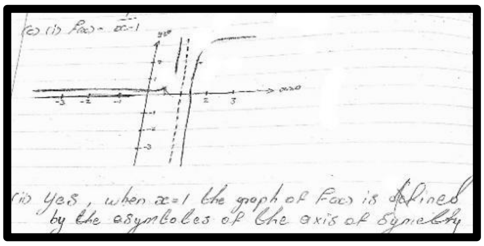
L12’s solution to question 1(b)



Sub Question1(c) on sketching the graph of *f(x)* = $\frac{1}{x-1}$ and using the graph to find $\lim_{x\to 1}\frac{1}{x-1}$ and $\lim_{x\to\infty}\frac{1}{x-1}$ was intended to check the learners’ knowledge of functions and graphs and their ability to relate this knowledge to the limit concept. In addressing the above question**,** this group of ten learners, (ce1) (10), displayed intrinsic calculus misconceptions by failing to sketch the correct graph due to deficiencies in the knowledge of functions concerning the range and domain of hyperbolic functions as well as their asymptotes.

Figure 6.

L29’s solution to question 1(c)



In Figure 6 above, the learner, L29, correctly identified $x = 1$ and *y=0* as asymptotes of the graph but there was no consistency in applying the concept of an asymptote as one part of the graph is shown crossing y = 0, which had been given as an asymptote. The concepts of domain and range were not mastered by the learner in earlier grades. Hence, the knowledge gaps on extrinsic calculus concepts such as functions impacted negatively on the learner’s ability to use graphs to address questions on limits.

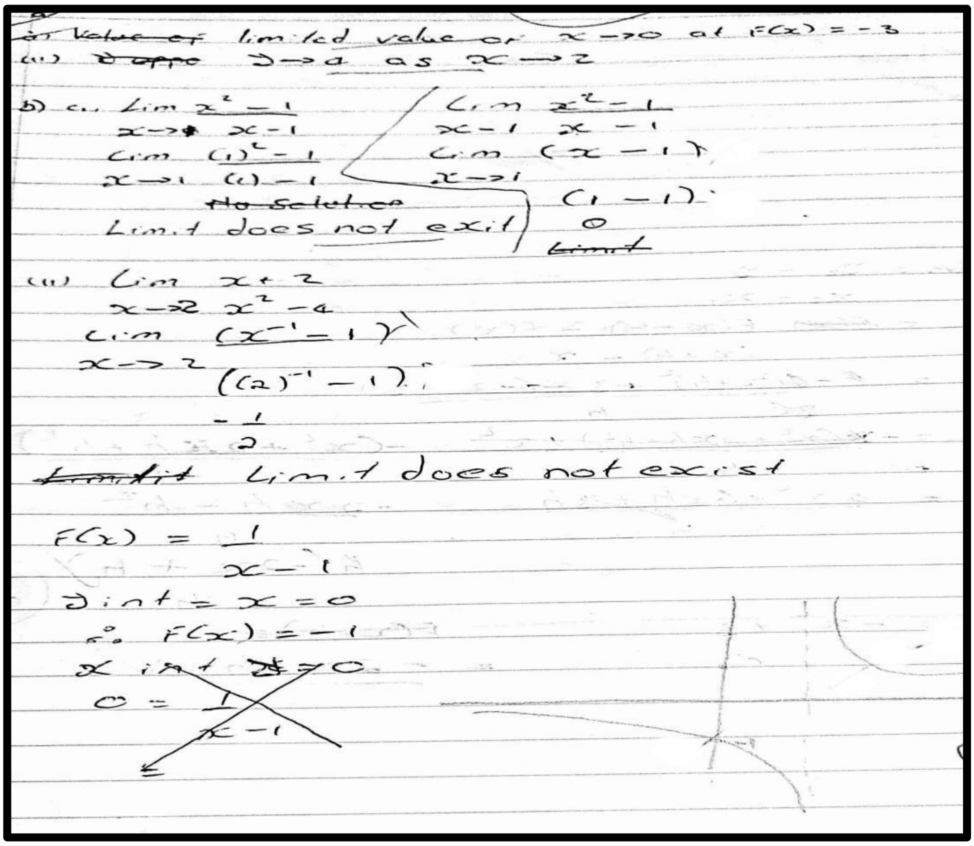
On this same question, another group of fifteen learners, (ci3) (15), displayed intrinsic calculus misconceptions on limits and discontinuity of functions at given points. These learners were able to sketch the correct graph of *f(x)* = $\frac{1}{x-1}$ but still could not use the correct graph to find limits due to knowledge gaps on continuity and discontinuity of functions at given points. Some learners in this group concluded *f(x)* is defined at x =1, since x= 1 is the asymptote of the graph*,* as L29 did above in Figure 6. For L29, the existence of the asymptote of the function implies *f(1)* exists. This is a conceptual error **(s1),** demonstrating a lack of understanding of the learner's concept of an asymptote.

In the next section, we present scripts of the students who were selected of interviews, followed by interviews questions and responses. Five learners L8, L9, L10, L11 and L12, were requested to clarify their written solutions orally to check their level of understanding of the limit concept.

Learner 8's misconceptions on the limit concept

Figure 7.

L8's solution to question 1


a) ~~Value of~~ limited value of x→0 at f(x) = -3
(ii) ~~appro~~ y→4 as x→2
b) (i) $\lim_{x\to 1} \frac{x^2-1}{x-1}$
$\lim_{x\to 1} \frac{(1)^2-1}{(1)-1}$
~~No solution~~
Limit does not exist
$\lim \frac{x^2-1}{x-1}$
$\lim_{x\to 1} (x-1)$
$(1-1)$
0
~~Limit~~
(ii) $\lim_{x\to 2} \frac{x+2}{x^2-4}$
$\lim_{x\to 2} (x^{-1}-1)$
$((2)^{-1}-1)$
$-\frac{1}{2}$
~~Limit~~ Limit does not exist
$f(x) = \frac{1}{x-1}$
y int = x = 0
∴ f(x) = −1
x int y = 0
$0 = \frac{1}{x-1}$

Researcher: *Can you explain more on the answers you gave on meaning of these statements* $\lim_{x\to 0} \frac{x^2-3x}{x}$ *= - 3 and* $\frac{x^2-4}{x-2}$ *→ 4 as x→ 2?*

L8: *limit means at the end so at the limit when x is goes to zero the function equals -3 and f(x)=y so in the second when y goes to 4 when x goes to 2.*

Researcher: *Please explain how you arrived at the answers that you got in 1b(i) &(ii)*

L8: *You substitute the value of x to get answer of limit. When I substitute x=1 in b (i)it is undefined, and limit does not exist. Even in b(ii) the limit does not exist because when x=2 we divide by zero and it is undefined.*

Researcher: *In b(I, explain this statement* $\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}(x-1)$*.*

L8 : *If you factorize* x^2-1 *=(x-1)(x-1) so* $\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}(x-1)$*.*

Researcher: *How many asymptotes does the graph f(x) =* $\frac{1}{x-1}$ *have?*

L8: *It has two asymptotes x=1 and y=0*

Researcher: *What are the x and y- intercepts of f(x) =* $\frac{1}{x-1}$

L8: *At y-int, x=0; we get y= -1 and at x-int, y=0, graph does not cross x-axis, y=0 is asymptote*

Researcher: *Is f(x) defined at x =1? Explain*

L8: *x=1 is asymptote of graph. graph cannot intersect x=1, we don't y-value*

Researcher: *How can you use your graph to find* $\lim_{x\to 1}\frac{1}{x-1}$ *and* $\lim_{x\to\infty}\frac{1}{x-1}$

L8: *It is not possible, x=1 is asymptote and ∞ is not a number which I can substitute to get limit*

From the responses, the learner *L8* had the misconception that the limit of a function is the value of the function for a given value of *x.* For L8, finding the limit of a function was equivalent to finding the value of the function by merely substituting the given values of *x*. This is incorrect since $\lim_{x\to a} f(x)$ can exist when *f(a)* is undefined.

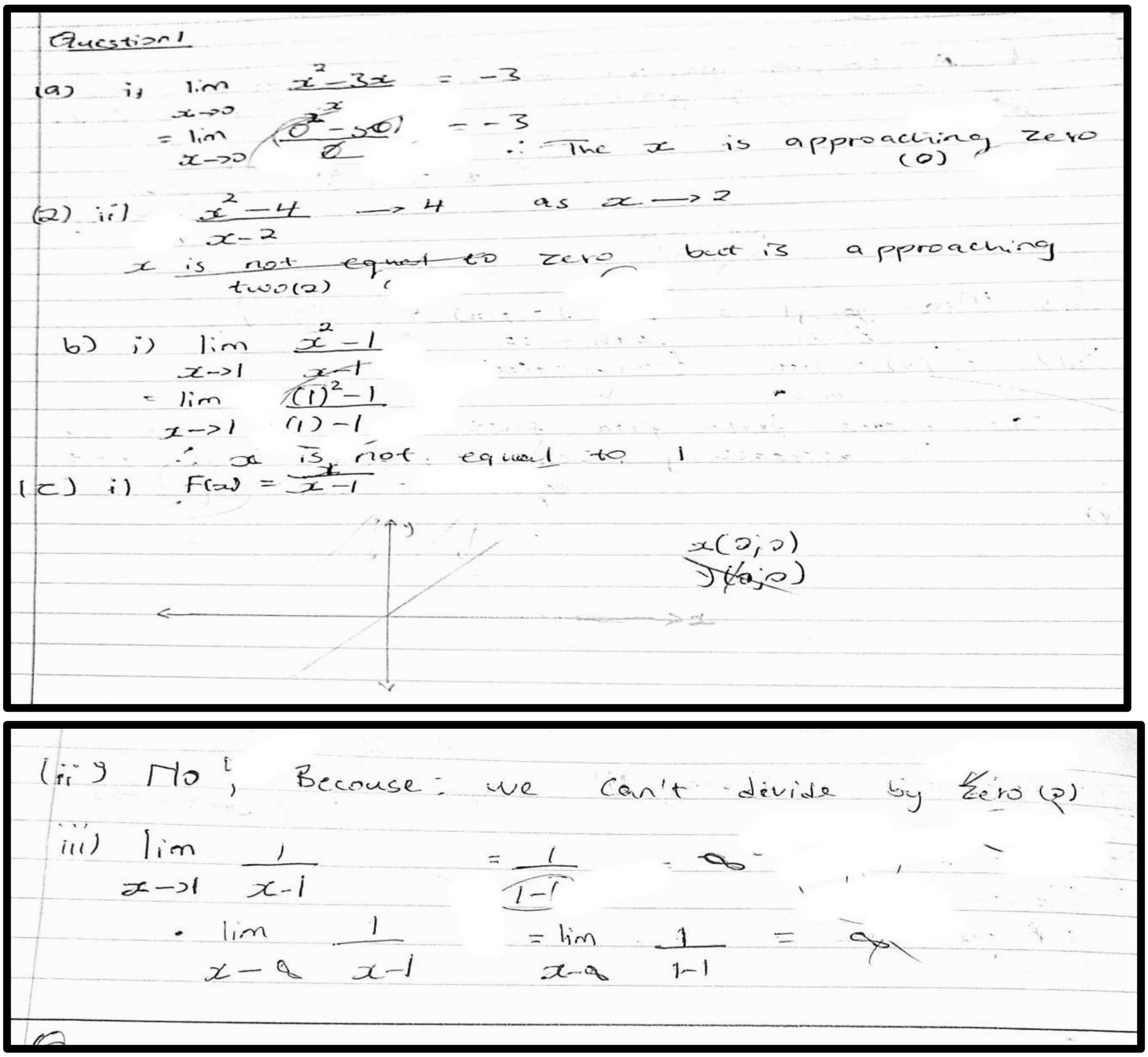
The learner explained limit to mean the value of *x* at the end, yet $\lim_{x\to a} f(x)$ means *"x is approaching and getting as close as is possible to a"* and this does not mean *x=a,* hence $\lim_{x\to a} f(x)$ should not be interpreted to mean *f(a).* This appeared to be a case of the learner choosing to rely on simple substitution strategies and simplification to address problems that required conceptual knowledge of the limit concept, and he ended up making incorrect conclusions.

The other misconception that $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$ $=\lim\limits_{x\to 1}(x\text{-}1)$, as argued by L8, is a result of poor a foundation on algebraic skills of factorization of a difference of two squares. This knowledge acts as the requisite prior knowledge for the learning and understanding the concept limit. If the learner had factorized correctly then his answer would have been $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$ $=\lim\limits_{x\to 1}(x+1)$ $=1+1=2$. *This* is evidence that this was an extrinsic calculus misconception on limits as it was caused by a flawed foundation in concepts that were learnt earlier.

Learner 9's misconceptions on the limit concept

Figure 8.

L9's written response to question 1



Researcher: *You substituted x by zero (0) in* $\lim\limits_{x\to 0}\frac{x^2-3x}{x}$*, why?*

L9 *x is approaching zero and at the end it is zero and when x =0,*

$\frac{x^2 - 3x}{x} = \frac{0^2 - 3(0)}{0}$

Researcher: *What do you understand by the = sign in the statement* $\lim_{x \to 0} \frac{x^2 - 3x}{x} = -3$?

L9: *: It means what is on the left is the same as what is on the right. But it is not correct because on the left they are dividing by zero which is not allowed.*

Researcher: *. Can you simplify the fraction* $\frac{x^2 - 3x}{x}$**?**

L9: $\frac{x^2 - 3x}{x} = \frac{x(x-3)}{x}$ = *x -3*

Researcher: *What is your answer to* $\lim_{x \longrightarrow 0} x - 3$?

L9: $\lim_{x \longrightarrow 0} x - 3$ *= 0 - 3 = - 3*

Researcher: *In* $\frac{x^2-4}{x-2} \rightarrow 4$ *as* $x \rightarrow 2$ *you explained that x is approaching 2 , what is happening to* $\frac{x^2-4}{x-2} \rightarrow 4$

L9: *L9. When x=2, we put 2 where there is x to get* $\frac{2^2-4}{2-2} \rightarrow 4$

Researcher: *Please explain how you arrived at the answer that you go in 1b (i).*

L9: *Substitute the value of x into expression to get answer.*

Researcher: *How many asymptotes does the graph f(x) =* $\frac{1}{x-1}$ *have?*

L9: *There is no asymptote, but I am not sure*

Researcher: *What are the x and y- intercepts of f(x) =* $\frac{1}{x-1}$

L9: *For x-int, y=0 and y-int ,x=0 ; the point is (0;0)*

Researcher: *Is f(x) defined at x =1 ? Explain*

L9: *When x=1 ,we divide by zero and this is not allowed; f(x) is not defined.*

Researcher: *How can you use your graph to find* $\lim_{x \to 1} \frac{1}{x-1}$ *and* $\lim_{x \to \infty} \frac{1}{x-1}$

L9: *If we put x=1, we divide by zero to get ∞*

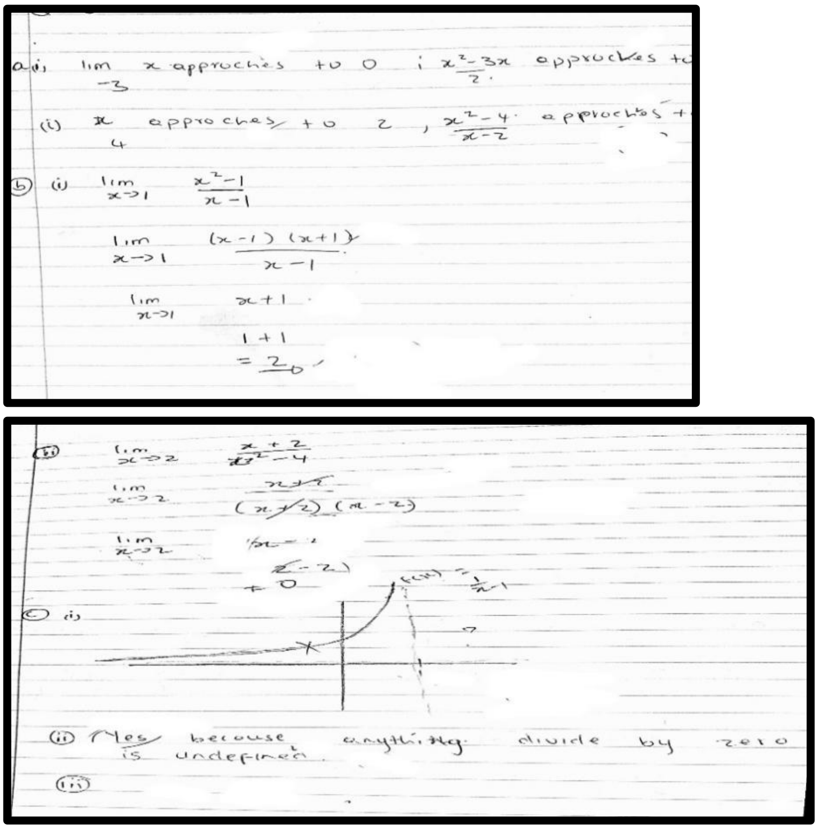
For this learner, L9, there was a conceptual error (s1), as the learner failed to grasp the concept of a limit where in $\lim_{x \to a} \frac{x^2 - 3x}{x}$, means *x-approaches and gets very close to zero but x≠0,* which means while $\lim_{x \to a} f(x)$ exists *f(0)* is undefined. The learner, L9*,* found the value of the function at the given x-value points. L9 had the misconception that the limit of a function was the value of the function for a given value of *x.* This is incorrect since $\lim_{x \to a}$ *f(x)* can exist when *f(a)* is undefined.

Learners *L9* failed to resolve the problem of finding limits using the graph because his knowledge of graphs of hyperbolic functions was flawed, particularly in relation to how asymptotes are determined and the general shape of the graph. The question on the number of asymptotes of *f(x) =* $\frac{1}{x-1}$ was intended to identify flaws in the learners' knowledge of algebraic functions in general and hyperbolic functions specifically. L9 drew a straight line for the graph of f*(x) =* $\frac{1}{x-1}$, an indication that the learner had a very weak understanding of functions. This impacts negatively on his attempts to learn to understand calculus concepts.

Learner 10's misconceptions on the limit concept

Figure 9.

L10's written answers to question 1



Researcher: *Can you explain more on the answers you gave on meaning of these statements?*

$\lim_{x \to 0} \frac{x^2 - 3x}{x} = -3$ *and* $\frac{x^2-4}{x-2} \to 4$ *as* $x \to 2$

L10: *Limit means when x gets close to zero, that is approaching, then* $\frac{x^2 - 3x}{x}$ *gets close to -3 .*

Researcher: *Please explain how you arrived at the answer that you go in 1b (i) &(ii).*

L10: *First you simplify the given expression, then you substitute the value of x at the end to get answer.*

Researcher: *You wrote* $\frac{\boldsymbol{x+2}}{(\boldsymbol{x-2})(\boldsymbol{x+2})}$ **=** *x – 2, may you please explain why?*

L10: *You cannot write* $\frac{1}{x-2}$ *because to find limit we end up dividing by zero which is not allowed.*

Researcher: *Is it correct to write* $\frac{1}{x-2} = x - 2$ *and* $\frac{1}{5} = 5$ *?*

L10: $\frac{1}{5} \neq 5$ *but if we write* $\frac{1}{x-2}$ *we cannot find limit at x =2*

Researcher: *How many asymptotes does the graph f(x) =* $\frac{1}{x-1}$ *have?*

L10: *Two asymptotes x=1 and y=0*

Researcher: *What are the x and y- intercepts of f(x) =* $\frac{1}{x-1}$

L10: *y-int, y=1, graph does not cross x-axis; there is no x-intercept*

Researcher: *Is f(x) defined at x =1? Explain*

L10: *Anything divide by zero is undefined; it means f(x) is undefined at x=1*

Researcher: *How can you use your graph to find* $\lim_{x \to 1} \frac{1}{x-1}$ *and* $\lim_{x \to \infty} \frac{1}{x-1}$

L10: *Graph cannot give answer, if we substitute x=1, answer is undefined you cannot substitute for ∞.*

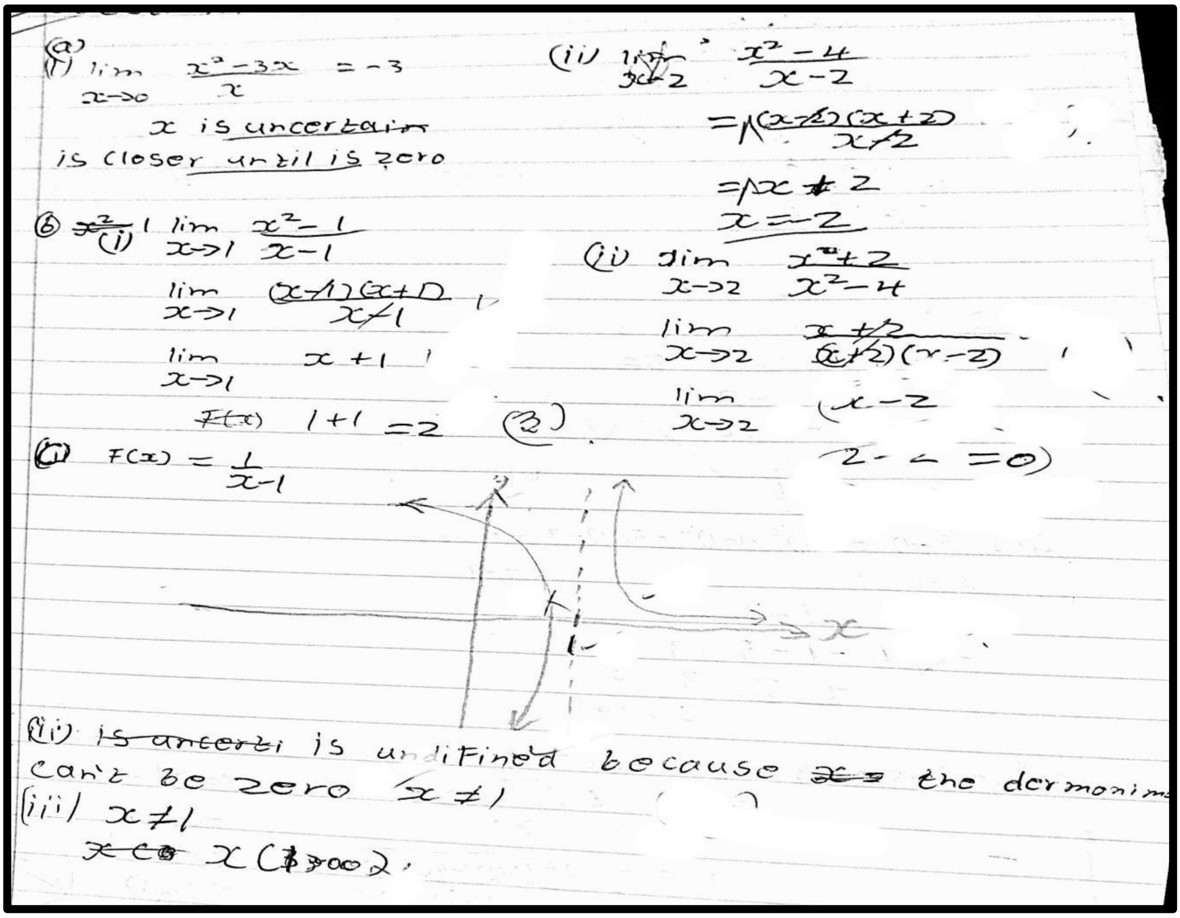
L10 appeared to understand that both *x* and *f(x)* values are values which they *approach* as opposed to values which they *are equal to* when he explained statements on limits. However, when required to find the limits, the learner struggled to identify cases where the limit did not exist because his understanding is limited to computing the final answer by substitution, hence when presented with $\lim_{x \to \infty} \frac{1}{x-1}$, the method of substitution failed and when the substitution gave a denominator of zero as in $\lim\limits_{x \to 1} \frac{1}{x-1}$, the learner failed to proceed. This is evidence of an instrumental understanding of the limit concept.

Secondly L10 had not mastered the concept continuity and discontinuity of functions at given points, he failed to find $\lim\limits_{x \to 2} \frac{1}{x-2}$ because he relied on simple strategies of substitution and simplifying without having a deeper understanding of the concept in question.

Learners L10 failed to resolve the problem of finding limits using the graph because his knowledge of graphs of hyperbolic functions was flawed, particularly in relation to how asymptotes are determined and the general shape of the graph. The learner drew the graph of f*(x)* = $\frac{1}{x-1}$ with one branch like an exponential graph, an indication that the learner could be confusing exponential and hyperbolic functions. Once the learner had a flawed understanding of the function in question, then it became impossible for the learner to utilize this incorrect knowledge on functions to resolve the question on limits of the same function that he did not understand. This prior misconception affected the learners' ability to learn new concepts on limits.

Figure 10.

L11`s written answers to question 1



Researcher: *Can you explain more on the answers you gave on meaning of* these statements

$\lim_{x \to 0} \frac{x^2 - 3x}{x}$ = - 3 and $\frac{x^2-4}{x-2} \to 4$ as $x \to 2$

L11: *We don't know the value of x but it is close to zero, when it becomes zero the answer is -3.*

L11: *For $\frac{x^2-4}{x-2}$ we factorise and simplify to find value of x and from x+2, x= -2*

Researcher: *Why did you find x ?*

L11: *This statement $\frac{x^2-4}{x-2} \to 4$ as $x \to 2$* is saying *x = 2, but x+2=0 gives x= -2.*

Researcher: *Please explain how you arrived at the answer that you go in 1b (i) &(ii).*

L11: *I simplify expression then substitute the number for x at the end to get limit.*

Researcher: *You wrote $\frac{\boldsymbol{x+2}}{(\boldsymbol{x-2})(\boldsymbol{x+2})}$* **=** *x – 2, may you please explain why?*

L11: *After dividing by x+2 , we are left with x -2 ; it is not divided*

Researcher: *Is it correct to write $\frac{1}{x-2} = x - 2$ and $\frac{1}{5} = 5$?*

L11: *$\frac{1}{5} \neq 5$, the first is a fraction and 5 is a whole number but if we replace x with 2 in $\boldsymbol{\lim_{x \to 2}} \frac{1}{x-2}$ we get zero in denominator. We do not divide by zero.*

Researcher: *How many asymptotes does the graph f(x) = $\frac{1}{x-1}$ have?*

L11: *Two asymptotes x=1 and y=0*

Researcher: *What are the x and y- intercepts of f(x) = $\frac{1}{x-1}$?*

L11: *I did not find the intercepts*

Researcher: *Is f(x) defined at x =1? Explain*

L11: *f(x) is undefined at x=1, we cannot divide by zero.*

Researcher: *How can you use your graph to find $\lim_{x \to 1} \frac{1}{x-1}$ and $\lim_{x \to \infty} \frac{1}{x-1}$*

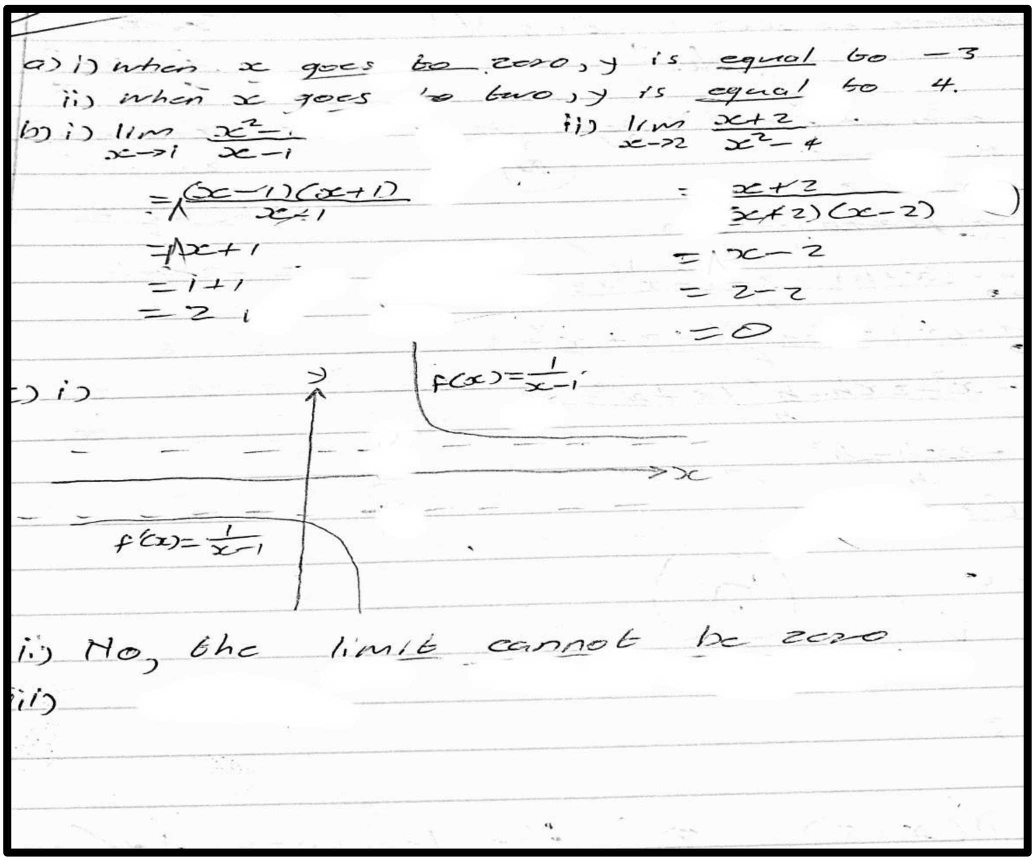
L11 *X≠1, we can not get limit, x is between 1 and ∞*

From these responses, L11 had the same misconception as L8 and L9, that the limit of a function is the value of the function for a given value of *x.* L11 did not understand the symbolism employed here for limits $\frac{x^2-4}{x-2}$ → *4 as x→ 2*. There also appeared to be confusion between an algebraic expression and an equation as the learner treated an expression $\frac{x^2-4}{x-2}$ as an equation. This is an example of a weak foundation in algebra affecting the learning of new concepts on limits resulting in extrinsic calculus misconceptions.

Secondly, L11 had not mastered the concept continuity and discontinuity of functions at given points, as he failed to find $\lim\limits_{x\to 2}\frac{1}{x-2}$ by relying on simple strategies of substitution and simplifying without having a deeper understanding of the concept in question. The learner failed to proceed because substitution gave a denominator of zero as the function is undefined at x=2, but this did not imply the non-existence of the limit.

Figure 11.

L12`s written answers to question 1


a) i) when x goes to zero, y is equal to -3
ii) when x goes to two, y is equal to 4.
b) i) $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$
$= \frac{(x-1)(x+1)}{x-1}$
$= x+1$
$= 1+1$
$= 2$
ii) $\lim\limits_{x\to 2}\frac{x+2}{x^2-4}$
$= \frac{x+2}{(x+2)(x-2)}$
$= x-2$
$= 2-2$
$= 0$
c) i)

ii) No, the limit cannot be zero
iii)

The learner, L11, failed to resolve the problem of finding limits using the graph because his knowledge of graphs of hyperbolic functions was flawed, particularly in relation to how asymptotes are determined and the general shape of the graph. L11 made the error of having one branch of their graph of f*(x)* = $\frac{1}{x-1}$ intersecting with an asymptote, y=0, an indication that their understanding of an asymptote is flawed, and this prior misconception affected the learners' ability to learn new concepts on limits.

Researcher: *Can you explain more on the answers you gave on meaning of these statements*

$\lim\limits_{x \to 0} \frac{x^2 - 3x}{x} = -3$ *and* $\frac{x^2-4}{x-2} \to 4$ *as* $x \to 2$

L12: *When x takes value of zero, y takes the value of – 3 and in (ii) when x takes the value of 2, y takes the value of 4*

Researcher: *Please explain how you arrived at the answer that you go in 1b (i) &(ii).*

L12: *You simplify the given expression then substitute x by the given number to get limit.*

Researcher: *You wrote* $\frac{\boldsymbol{x+2}}{(\boldsymbol{x-2})(\boldsymbol{x+2})}$ **=** *x – 2, may you please explain why?*

L12: *Only x -2 is left after cancelling equal factors at the top and bottom.*

Researcher: *Is it correct to write* $\frac{1}{x-2} = x - 2$ *and* $\frac{1}{5} = 5$

L12: *It is not correct* $\frac{1}{5} \neq 5$ *, it means* $\frac{1}{x-2} \neq x - 2$ *but I cannot get answer when I substitute x=2 in* $\frac{1}{x-2}$*.*

Researcher: *You wrote* $\lim\limits_{\boldsymbol{x \to 1}} \frac{\boldsymbol{x^2-1}}{\boldsymbol{x-1}} = \frac{(x+1)(x-1)}{x-1}$ *= x+1= 1+1=2, why did you omit* $\lim\limits_{\boldsymbol{x \to 1}}$ in next two steps?

L12: *I was simplifying, I used x= 1 at the end to find final answer.*

Researcher: *How many asymptotes does the graph f(x) =* $\frac{1}{x-1}$ *have?*

L12: *There are two asymptotes at y= -1 and y=1*

Researcher: *What are the x and y- intercepts of f(x) =* $\frac{1}{x-1}$

L12: *There is no x-intercept, y-int is y= -1*

Researcher: *Is f(x) defined at x =1? Explain*

L12: *No, you cannot have zero dividing in the limit.*

Researcher: *How can you use your graph to find* $\lim\limits_{x\to 1}\frac{1}{x-1}$ *and* $\lim\limits_{x\to\infty}\frac{1}{x-1}$

L12: *If x =1 , we get* $\frac{1}{1-1}$*, have zero at the bottom and there is no number for ∞.*

From their responses, L12 had the same misconception as learners L8,L9 and L11*,* namely that the limit of a function is the value of the function for a given value of *x* , that $\lim\limits_{x\to a} f(x) = f(a)$. This is a misconception because $\lim\limits_{x\to a} f(x)$ can exist in cases where $f(a)$ is undefined as in $\lim\limits_{x\to 1}\frac{x^2-1}{x-1}$.

Secondly, *L12* had not mastered the concept of continuity and discontinuity of functions at given points, He failed to find $\lim\limits_{x\to 2}\frac{1}{x-2}$ because the function is not defined at x=2. The method of substitution and simplifying without having a deeper understanding of the concept in question failed in cases where there is a discontinuity. It is because of the instrumental understanding of the limit concept that L12 omitted $\boldsymbol{\lim\limits_{x\to a}}$ in the procedures leading to the answers, as his focus was obtaining the correct answer. The omission of $\lim\limits_{x\to 1}$ led to a false generalization in this statement $\lim\limits_{x\to 1}\frac{\boldsymbol{x^2-1}}{\boldsymbol{x-1}} = \frac{(x+1)(x-1)}{x-1}$ = *x+1= 1+1=2* as it is not always true that x+1 =2 regardless of the value of x. Therefore, the correct statement should be $\lim\limits_{x\to 1}$*(x+1) =2*. L12 did not seem to appreciate the importance of utilizing the correct symbolism for limits which reflects a lack of understanding of the concept of limit itself.

L12 failed to resolve the problem of finding limits using the graph because his knowledge of graphs of hyperbolic functions was flawed, particularly in relation to how asymptotes are determined and the general shape of the graph. The learner gave two asymptotes parallel to the x-axis, which is an indication of a poor mastery of knowledge on hyperbolic functions. Once the learner's existing knowledge of functions was flawed, it then created room for other related misconceptions when the learner had to build new concepts of limits based on a weak mathematical foundation.

CONCLUSION and RECOMMENDATIONS

The research findings here confirms that several learner conceptual errors and misconceptions when learning calculus concepts were a product of a weak foundation firstly in extrinsic calculus concepts such as functions and related subfunctions, algebraic skills such as substitution, multiplication, factorization, and laws of exponents. Secondly, some errors and misconceptions reflect learners having difficulties with intrinsic calculus concepts such as the limit concept and related symbolism and continuity and discontinuity of functions at given points. The results also agree with researchers who argued that teaching which emphasizes drill and practice leads to procedural knowledge only (Hiebert & Lefevre, 1986; Machaba, 2018; Orton, 1983; Vinner, 1989). Procedural knowledge is instrumental and shallow, and there is a risk that learners with this kind of knowledge will regard the mastery of computational skills as the main objective of learning calculus, thereby losing the essence of learning mathematical concepts (Aspinwall and Miller, 1997; Bezuidenhout,2001; Davis & Vinner, 1986). Effort should be made by educators to show linkages where concepts are related, such as in functions and calculus, to ensure that learners develop a relational understanding of these concepts. Related concepts are treated in isolation the net effect is that students tend to adopt and retain incorrect mathematical constructs (Cooley et al., 2007; Orton, 1983; Selden et al., 1994). As a result, there could be conceptual errors (s1), and generalization or transfer errors (s2), when learners are confronted with broader contexts in which they have to apply these concepts.

The study showed that there are intrinsic calculus misconceptions (ci) in the learners written solutions and oral responses to interview questions. In these misconceptions, learners made systematic errors on concepts within calculus such as limits and algebraic manipulations and graphical application of calculus. Evidence from samples of learner solutions indicated that learners had only mastered an instrumental understanding of the calculus concepts that were taught and had no problems with executing procedures of limits but lacked conceptual understanding of the calculus concepts with which they were dealing. According to Skemp (1976, p. 20), “instrumental understanding is manifested when learners know rules and formulae and have the ability to use them without reason, not knowing where those rules and formulae come from”. This lack of conceptual knowledge or relational understanding of limits resulted in a wide range of systematic errors from conceptual errors (s1) to incorrect hypotheses being used to learn new calculus concepts (s5). Therefore, as a recommendation on intrinsic calculus misconceptions(ci), educators should adopt teaching methods which give equal attention to both the procedural and conceptual knowledge of limits of functions during the teaching-learning process, to ensure that learners know both *what* to do and *why.*

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APPENDIX

Limits Test

(a) Explain in words the meaning of these statements

(i) $\lim_{x \to 0} \frac{x^2 - 3x}{x}$ *= - 3*

(ii) $\frac{x^2-4}{x-2} \to 4$ *as* $x \to 2$

(b) Find the limits where they exist

(i) $\lim_{x \to 1} \frac{x^2-1}{x-1}$

(ii) $\lim_{x \to 2} \frac{x+2}{x^2-4}$

(c) (i) Sketch the graph of f(x) = $\frac{1}{x-1}$

(ii) Is f(x) defined at x =1 ? Explain

(iii) Use the graph to find $\lim_{x \to 1} \frac{1}{x-1}$ *and* $\lim_{x \to \infty} \frac{1}{x-1}$